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Correlation Inequalities for the Truncated Two-Point Function of an Ising Ferromagnet

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We establish the following new correlation inequalities for the truncated two-point function of an Ising ferromagnet in a positive external field: $\langle\sigma_j;\sigma_l\rangle^T \geqslant \langle\sigma_j;\sigma_k\rangle^T\langle\sigma_k;\sigma_l\rangle^T$, and $\langle\sigma_j;\sigma_l\rangle^T \leqslant \sum_{k\in K}\langle\sigma_j;\sigma_k\rangle^T\langle\sigma_k\sigma_l\rangle$, where K is any set of sites which separates j from l. The inequalities are also valid for the pure phases with zero magnetic field at all temperatures. Above the critical temperature they reduce to known inequalities of Griffiths and Simon, respectively.

KEY WORDS: Correlation inequalities; Ising model.

1. INTRODUCTION

We consider an Ising model with spins $\sigma_i = \pm 1$, on sites $i = 1, \ldots, N$, and Hamiltonian

$$-H = \sum_{1\leqslant i<j\leqslant N} J_{ij}\sigma_i\sigma_j + \sum_{i=1}^{N} h_i\sigma_i \tag{1.1}$$

with $J_{ij} \geqslant 0$, $h_i \geqslant 0$. The partition function $Z = 2^{-N}\sum_{\sigma_i=\pm 1} e^{-H}$ (where we have set the inverse temperature β equal to 1) and expectations are defined by

$$\langle\cdot\rangle = 2^{-N}\sum_{\sigma_i=\pm 1}(\cdot)e^{-H}/Z \tag{1.2}$$

We denote the truncated two-point function by

$$\langle\sigma_k;\sigma_l\rangle^T \equiv \langle\sigma_k\sigma_l\rangle - \langle\sigma_k\rangle\langle\sigma_l\rangle \tag{1.3}$$

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Theorem 1. Let $J_{ij} \geqslant 0$, $h_i \geqslant 0$ in (1.1). Then

$$\langle \sigma_j; \sigma_l \rangle^T \geqslant \langle \sigma_j; \sigma_k \rangle^T \langle \sigma_k; \sigma_l \rangle^T \tag{1.4}$$

We will refer to each pair of sites $\{i, j\}$ for which $J_{ij} > 0$ as a *bond*. A *path from k to l* is a collection of bonds $b_1, \ldots, b_s$ such that $k \in b_1$, $l \in b_s$, and b_i and b_{i+1} have a site in common for $i = 1, \ldots, s-1$. A collection of sites *K separates j from l* if every path from j to l contains an element of K.

Theorem 2. Let $J_{ij} \geqslant 0$ and $h_i \geqslant 0$ in (1.1) and let K separate j from l. Then

$$\langle \sigma_j; \sigma_l \rangle^T \leqslant \sum_{k \in K} \langle \sigma_j; \sigma_k \rangle^T \langle \sigma_k \sigma_l \rangle \tag{1.5}$$

Remark 1. Theorem 1 is reminiscent of the inequality $\langle \sigma_j \sigma_l \rangle \geqslant \langle \sigma_j \sigma_k \rangle \langle \sigma_k \sigma_l \rangle$ due to Griffiths.[1,2] Theorem 2 resembles Simon's inequality[3]:

$$\langle \sigma_j \sigma_l \rangle \leqslant \sum_{k \in K} \langle \sigma_j \sigma_k \rangle \langle \sigma_k \sigma_l \rangle$$

It would be desirable to prove an inequality of the form $\langle \sigma_j; \sigma_l \rangle^T \leqslant \sum_{k \in K} \langle \sigma_j; \sigma_k \rangle^T \langle \sigma_k; \sigma_l \rangle^T$, however Baker and Bessis[4] have strong evidence (although no proof) that this cannot hold for the two-dimensional Ising model.

Remark 2. Theorem 1 and the work of Baker and Bessis[4] show that if we assume for $T < T_c$ that $\langle \sigma_0; \sigma_r \rangle^T$ decays like e^{-mr}/r^b, then $b \geqslant 0$.

We employ Griffiths' "ghost spin" method[2] which relates the model (1.1) to a new model (indicated by primes) which has an extra spin σ_0 (the "ghost" spin) and *zero external field*. Specifically, the new Hamiltonian is

$$-H' = \sum_{0 \leqslant i < j \leqslant N} J'_{ij} \sigma_i \sigma_j \tag{1.6}$$

where $J'_{ij} = J_{ij}$ for $i \neq 0$ and $J'_{0j} = h_j$. Correlation functions are defined in the usual way. We have

$$\langle \sigma_{i_1} \ldots \sigma_{i_n} \rangle = \begin{cases} \langle \sigma_{i_1} \ldots \sigma_{i_n} \rangle' & \text{if } n \text{ is even} \\ \langle \sigma_0 \sigma_{i_1} \ldots \sigma_{i_n} \rangle' & \text{if } n \text{ is odd} \end{cases}$$

Theorems 1 and 2 are contained in the following two results.

Theorem 3. Let $J'_{ij} \geqslant 0$ in (1.6). Then

$$\langle \sigma_j \sigma_l \rangle' - \langle \sigma_j \sigma_0 \rangle' \langle \sigma_l \sigma_0 \rangle' \geqslant (\langle \sigma_j \sigma_k \rangle' - \langle \sigma_j \sigma_0 \rangle' \langle \sigma_k \sigma_0 \rangle')(\langle \sigma_k \sigma_l \rangle' - \langle \sigma_k \sigma_0 \rangle' \langle \sigma_l \sigma_0 \rangle') \tag{1.7}$$

Theorem 4. Let $J'_{ij} \geqslant 0$ in (1.6) and let K separate j from l. Then

$$\langle\sigma_j\sigma_l\rangle' - \langle\sigma_j\sigma_0\rangle'\langle\sigma_l\sigma_0\rangle' \leqslant \sum_{k\in K} (\langle\sigma_j\sigma_k\rangle' - \langle\sigma_j\sigma_0\rangle'\langle\sigma_k\sigma_0\rangle')\langle\sigma_k\sigma_l\rangle' \quad (1.8)$$

We conclude this section by giving an outline of the rest of the paper. Graphical methods are introduced in Section 2, and in Section 3 they are used to prove Theorems 3 and 4.

2. GRAPHICAL METHODS

We follow the notation of Aizenman[5] but note that these graphical expansions have been used previously by Kelly and Sherman,[6] Giffiths, Hurst, and Sherman,[7] Newman,[8] and Simon.[3] A key lemma for our analysis first appeared in the paper of Griffiths *et al.*[7]

Earlier, Griffiths[1,2,9] and Fisher[10] used distinct, although similar, graphical expansions.

For the remainder of the paper we will consider only the Hamiltonian (1.6) and the primes will therefore be dropped from the notation.

Write

$$Z = 2^{-(N+1)} \sum_{\sigma_i=\pm 1} \prod_{0\leqslant i<j\leqslant N} e^{J_{ij}\sigma_i\sigma_j}$$

and for each bond expand the exponential

$$e^{J_b\sigma_i\sigma_j} = \sum_{n_b=0}^{\infty} \frac{J_b^{n_b}}{n_b!} (\sigma_i\sigma_j)^{n_b}$$

After averaging over the $\{\sigma_i\}$ we obtain

$$Z = \sum_{\partial\mathbf{n}=\phi} W(\mathbf{n}) \quad (2.1)$$

where $\mathbf{n}$ is an assignment of positive integers to the bonds (we regard these integers as fluxes),

$$W(\mathbf{n}) = \prod_b \frac{J_b^{n_b}}{n_b!} \qquad \text{and} \qquad \partial\mathbf{n} = \left\{ i \mid \prod_{b\ni i} (-1)^{n_b} = -1 \right\}$$

i.e., the set of sites where the net flux is odd. We call the set $\partial\mathbf{n}$ the boundary of $\mathbf{n}$ and refer to its elements as sources.

If we apply this procedure to correlation functions we find

$$\langle\sigma_K\rangle = \sum_{\partial\mathbf{n}=K} W(\mathbf{n}) \Big/ \sum_{\partial\mathbf{n}=\phi} W(\mathbf{n}) \quad (2.2)$$

(where $\sigma_K = \prod_{k\in K}\sigma_k$).

We will sometimes refer to an assignment of fluxes as a *graph*. Call $\mathbf{s}$ a subgraph of $\mathbf{n}$ if $0 \leqslant s_b \leqslant n_b$ for all b and write $\mathbf{s} \leqslant \mathbf{n}$.

A very useful tool is provided by the following result of Griffiths *et al.*[7]

Lemma 1. Let V_1 and V_2 be sets of sites. Then

$$\sum_{\substack{\partial \mathbf{n}_1 = V_1 \\ \partial \mathbf{n}_2 = V_2}} W(\mathbf{n}_1) W(\mathbf{n}_2) = \sum_{\substack{\partial \mathbf{n}_1 = V_1 \Delta V_2 \\ \partial \mathbf{n}_2 = \phi}}' W(\mathbf{n}_1) W(\mathbf{n}_2) \tag{2.3}$$

where the primed summation has the restriction that $\mathbf{n}_1 + \mathbf{n}_2$ has a subgraph $\mathbf{s}$ with $\partial \mathbf{s} = V_2$. ($\Delta$ indicates the usual symmetric difference.)

This lemma was used in Ref. 5 to give a probabilistic content to correlation functions and various inequalities. The intuition gained by this procedure makes the method considerably more powerful: see the results established there.

The *connected cluster of j in* $\mathbf{n}_1 + \mathbf{n}_2$, denoted $C_{\mathbf{n}_1+\mathbf{n}_2}(j)$, is the collection of sites that may be reached from j via bonds on which $\mathbf{n}_1 + \mathbf{n}_2 > 0$. The *connected cluster of bonds of j in* $\mathbf{n}_1 + \mathbf{n}_2$, $C^b_{\mathbf{n}_1+\mathbf{n}_2}(j)$, is the set of all bonds that may be reached from j via bonds on which $\mathbf{n}_1 + \mathbf{n}_2 > 0$. By definition, all bonds in $C^b_{\mathbf{n}_1+\mathbf{n}_2}(j)$ have $\mathbf{n}_1 + \mathbf{n}_2 > 0$. Note that when $C^b_{\mathbf{n}_1+\mathbf{n}_2}(j)$ is specified, the additional information that the adjacent bonds are unoccupied is implied. The set of bonds formed by the addition of these unoccupied bonds to $C^b_{\mathbf{n}_1+\mathbf{n}_2}(j)$ we denote $\bar{C}^b_{\mathbf{n}_1+\mathbf{n}_2}(j)$, and refer to it as the *augmented cluster of bonds of j in* $\mathbf{n}_1 + \mathbf{n}_2$. We will say $\mathbf{n}_1 + \mathbf{n}_2$ *has a path from j to k* if there exists a path from j to k, $b_1, \ldots, b_s$ for which $n_{1b_i} + n_{2b_i} > 0$, $i = 1, \ldots, s$, that is if $k \in C_{\mathbf{n}_1+\mathbf{n}_2}(j)$.

The set of all bonds we will denote by $\mathbb{B}$.

For compactness, we introduce the following additional notation which is similar to that used by Newman.[8]

We denote $\sum_{\partial n = V} W(\mathbf{n})$ by (V). The partition function will then be denoted by (ϕ). Expressions such as

$$\sum_{\substack{\partial \mathbf{n}_1 = j,k \\ \partial \mathbf{n}_2 = \phi}}' W(\mathbf{n}_1) W(\mathbf{n}_2)$$

where the prime refers to the restriction that $C_{\mathbf{n}_1+\mathbf{n}_2}(j) \ni 0$, will be denoted by $(C(j) \ni 0 \mid \{j,k\}, \phi)$.

Lemma 1 may be used to derive an appealing expression for the truncated two-point function:

$$\begin{aligned}
\langle \sigma_j \sigma_k \rangle - \langle \sigma_j \sigma_0 \rangle \langle \sigma_k \sigma_0 \rangle &= \frac{1}{Z^2} \left((j,k)(\phi) - (j,0)(k,0) \right) \\
&= \frac{1}{Z^2} \left((j,k)(\phi) - (C(j) \ni 0 \mid \{j,k\}, \phi) \right) \\
&= \frac{1}{Z^2} \left((C(j) \not\ni 0 \{j,k\}, \phi) \right) \\
&= \langle \sigma_j \sigma_k \rangle \operatorname{Prob}(C(j) \not\ni 0 \mid \{j,k\}, \phi) \qquad (2.4)
\end{aligned}$$

where the probability is evaluated in the system of two independent flux variables with the specified sources.

The other tool we will need is the following simple corollary of the GKS inequalities.[1,2,6]

Lemma 2. Let $A \subseteq B$ be a set of bonds and denote by $\langle \cdot \rangle_A$, the Gibbs state obtained by setting all the interactions to zero in A^c. Then

$$\langle \sigma_j \sigma_j \rangle - \langle \sigma_j \sigma_k \rangle_A \geqslant 0$$

3. PROOFS OF THEOREMS 3 AND 4

3.1. Proof of Theorem 3

Since Z is positive, (1.7) is equivalent to showing the positivity of

$$\begin{aligned}
&(\phi)^2((j,l)(\phi) - (j,k)(k,l)) - (\phi)(l,0)((j,0)(\phi) - (j,k)(k,0))\\
&\quad \times (j,0)\big((\phi)(k,l)(k,0) - (l,0)(k,0)^2\big)\\
&= (\psi)^2((j,l)(\psi) - (C(k) \ni j \,|\, \{j,l\},\phi))\\
&\quad -(\phi)(l,0)((j,0)(\phi) - (C(k) \ni 0 \,|\, \{j,0\},\phi))\\
&\quad +(j,0)((\psi)(C(k) \ni 0 \,|\, \{l,0\},\phi) - (l,0)(C(k) \ni 0 \,|\, \phi,\phi))\\
&= (\phi)^2(C(k) \not\ni j \,|\, \{j,l\},\phi) - (\phi)(l,0)(C(k) \not\ni 0 \,|\, \{j,0\},\phi)\\
&\quad +(j,0)((l,0)(C(k) \not\ni 0 \,|\, \phi,\phi) - (\phi)(C(k) \not\ni 0 \,|\, \{l,0\},\phi))
\end{aligned} \tag{3.1}$$

where we have used Lemma 1 to get the first equality and added and subtracted $(j,0)(l,0)(\phi)^2$ to get the second one.

The first term in (3.1) may be written as

$$\begin{aligned}
&(\phi)^2(C(k) \not\ni j, C(j) \not\ni 0 \,|\, \{j,l\},\phi) + (\phi)^2(C(k) \not\ni j, C(j) \ni 0 \,|\, \{j,l\},\phi)\\
&\quad = (\phi)^2(C(k) \not\ni j, C(j) \not\ni 0 \,|\, \{j,l\},\phi) + (\phi)^2(C(k) \not\ni j \,|\, \{j,0\},\{l,0\})
\end{aligned}$$

Hence (3.1) is not smaller than

$$\begin{aligned}
&(\phi)^2(C(k) \not\ni 0 \,|\, \{j,0\},\{l,0\}) - (\phi)(l,0)(C(k) \not\ni 0 \,|\, \{j,0\},\phi)\\
&\quad + (j,0)(l,0)(C(k) \not\ni 0 \,|\, \phi,\phi) - (j,0)(\phi)(C(k) \not\ni 0 \,|\, \{l,0\},\phi)
\end{aligned} \tag{3.2}$$

In each term of (3.2) we have the same restriction $C_{\mathbf{n}_1+\mathbf{n}_2}(k) \not\ni 0$. We will evaluate the terms by conditioning on the event "the connected cluster of bonds of k in $\mathbf{n}_1 + \mathbf{n}_2$ is A"and then summing over the allowed class of A's.

We will denote by $(C^b(k) \equiv A \mid \phi, \phi)_A$ the sum

$$\sum_{\substack{\partial \mathbf{n}_1 = \phi \\ \partial \mathbf{n}_2 = \phi}}' W(\mathbf{n}_1) W(\mathbf{n}_2)$$

where the primed summation has the restrictions that the interaction has been set to zero in A^c and $C^b_{\mathbf{n}_1+\mathbf{n}_2}(k) \equiv A$. By $(\phi)_{\bar{A}^c}$ we mean $\sum''_{\partial \mathbf{n}_1 = \phi} W(\mathbf{n}_1)$, where the double primed summation has the restriction that the interaction is zero in $\bar{A}$. ($\bar{A}$ is the augmented cluster of bonds.)

(3.2) becomes

$$\sum_A' \left(C^b(k) \equiv A \mid \phi, \phi\right)_A (\phi)^2_{\bar{A}^c} (\phi)^2 (\langle j,0\rangle_{\bar{A}^c} \langle l,0\rangle_{\bar{A}^c} - \langle l,0\rangle \langle j,0\rangle_{\bar{A}^c} + \langle j,0\rangle\langle l,0\rangle - \langle j,0\rangle \langle l,0\rangle_{\bar{A}^c})$$

where the primed summation is over connected sets of bonds A such that k belongs to at least one of the bonds and 0 belongs to none of them,

$$= \sum_A' \left(C^b(k) \equiv A \mid \phi, \phi\right)_A (\phi)^2_{\bar{A}^c} (\phi)^2 (\langle j,0\rangle - \langle j,0\rangle_{\bar{A}^c})(\langle l,0\rangle - \langle l,0\rangle_{\bar{A}^c}) \tag{3.3}$$

(3.3) is positive by Lemma 2. ■

We state a corollary of the proof which will be used by the author elsewhere.[11]

Corollary. Let $J_{ij} \geqslant 0$ in (1.6). Then

$$(C(k) \not\ni 0 \mid \{j,k\}, \{k,l\})(\phi)^2 \geqslant (C(k) \not\ni 0 \mid \{j,k\}, \phi)(C(k) \not\ni 0 \mid \{k,l\}, \phi) \tag{3.4}$$

Proof. In obtaining (3.2) from (3.1) the term $(\phi)^2(C(k) \not\ni j, C(j) \not\ni 0 \mid \{j,l\}, \phi)$ was dropped. Hence, given (1.7) and (2.4) we have

$$(\phi)^2(C(j) \not\ni 0 \mid \{j,l\}, \phi) - (\phi)^2(C(k) \not\ni j, C(j) \not\ni 0 \mid \{j,l\}, \phi) \geqslant (C(k) \not\ni 0 \mid \{j,k\}, \phi)(C(k) \not\ni 0 \mid \{k,l\}, \phi)$$

Therefore $(\phi)^2(C(j) \not\ni 0,\ C(k) \ni j \mid \{j,l\}, \phi) \geqslant (C(k) \not\ni 0 \mid \{j,k\}, \phi)(C(k) \not\ni 0 \mid \{k,l\}, \phi)$. This is equivalent to (3.4) by Lemma 1. ■

3.2. Proof of Theorem 4

(1.8) is equivalent to

$$(\phi)(C(j) \not\ni 0 \mid \{j,l\}, \phi) \leqslant \sum_{k \in K} (C(j) \not\ni 0 \mid \{j,k\}, \phi)(k,l) \tag{3.5}$$

Note that

$$(\phi)(C(j) \not\ni 0 \mid \{j,l\},\phi) \leqslant \sum_{k \in K} (\phi)(C(j) \not\ni 0, C(j) \ni k \mid \{j,l\},\phi)$$

since the condition $\partial \mathbf{n}_1 = \{j,l\}$, $\partial \mathbf{n}_2 = \phi$ forces $\mathbf{n}_1 + \mathbf{n}_2$ to have a path from j to l and every such path contains at least one element of K.

Hence, using Lemma 1 it is enough to show the positivity of

$$(C(j) \not\ni 0 \mid \{j,k\},\phi)(k,l) - (C(j) \not\ni 0 \mid \{j,k\},\{k,l\})(\phi) \qquad (3.6)$$

for each k.

We condition on the event "$C^b_{\mathbf{n}_1+\mathbf{n}_2}(0) \equiv A$" and sum over A. (3.6) becomes

$$\sum_A{}' \left(C^b(0) \equiv A \mid \phi,\phi\right)_A (j,k)_{\bar{A}^c}(\phi)_{\bar{A}^c}(\phi)(\langle k,l\rangle - \langle k,l\rangle_{\bar{A}^c}) \qquad (3.7)$$

where the primed summation is over connected sets of bonds A such that 0 belongs to at least one of the bonds and j and k belong to none of them.

(3.7) is positive by Lemma 2. ■

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